

# Multiple solutions and numerical analysis to the dynamic and stationary models coupling a delayed energy balance model involving latent heat and discontinuous albedo with a deep ocean

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We study a climatologically important interaction of two of the main components of the geophysical system by adding an energy balance model for the averaged atmospheric temperature as dynamic boundary condition to a diagnostic ocean model having an additional spatial dimension. In this work, we give deeper insight than previous papers in the literature, mainly with respect to the 1990 pioneering model by Watts and Morantine. We are taking into consideration the latent heat for the two phase ocean as well as a possible delayed term. Non uniqueness for the initial boundary value problem, uniqueness under a non-degeneracy condition, and the existence of multiple stationary solutions are proved here. These multiplicity results suggest that an S-shaped bifurcation diagram should be expected to occur in this class of models generalizing previous Energy Balance Models (EBMs). The numerical method applied to the model is based on a finite volume scheme with nonlinear WENO reconstruction and Runge-Kutta TVD for time integration.

# 1. Introduction

This paper presents new contributions on the mathematical study of a climate model coupling atmosphere and ocean under a simplified formulation. Our main goal is to exhibit the possible multiplicity of solutions due to presence of an abruptly distributed albedo, such as it was formulated in terms of a discontinuous function by the climatologist M.I. Budyko (see [13]). Among the new effects considered with respect to previous mathematical treatments in the literature we consider here a positive latent heat for the ocean and a general memory term for the top ocean surface temperature. Moreover, we present here the numerical approximation of solutions by means of finite volume methods. We shall indicate also many other references on the mathematical treatment of this class of problems, in a survey style, trying to be useful in the necessary dialog between geophysical and mathematician experts.

Our model tries to understand the deterministic interactions between two of the main components of the climatic system. It is well known that detailed mathematical models of the atmosphere, the ocean and ice sheets are available (see, for instance, the proceedings of several meetings devoted to this topic, as it was the case of the references [33], [27], [28] and [12]). Nevertheless, investigating inherently transient phenomena with periods of 100 to 100,000 years is, of course, out of question for such sophisticated models. This is one reason why simpler models form useful tools in theoretical climatology. In addition, the mathematical treatment of such models is far to be obvious and requires the application of finer techniques of the mathematical and numerical analysis of nonlinear partial differential equations.

Our model takes in account, at least implicitly, the multiple spatial scales which arise in such complex coupling. Indeed, instead of considering the atmosphere temperature we shall work, as usual in the theory of EBM, with the averaged surface temperature on suitable spatial and time local scales. It is well known that in spite of its simplicity, this kind of averaged equations preserve a high sensitivity with respect to solar parameters. This is very useful for the study in very large time scales. Nevertheless, since the heat capacity of the ocean is so large, any departure from equilibrium in the ocean must have a fairly large effect on the thermodynamic state of the atmosphere. As for the ocean, although we can also simplify its modelling we must maintain the fact that cold water in a few localized regions at high latitudes sinks is distributed throughout the deep ocean by currents and slowly rises towards the surface. So, following [70], we maintain the ocean depth scale for the deep ocean and identify the ocean mixed layer with the averaged atmospheric surface. This type of models allows to find some explanations to the Glacial-Holocene transition (see [70]). The inclusion of some stochastic internal and external variations imperfectly known, as it is the case of solar luminosity variations, volcanic aerosols and  $CO_2$ , have already studied for the associated surface EBM (see, e.g. [56], [45], [23], [36]).

It is clear that more realistic ocean models can be also considered in order to investigate the interactions between time and space scales of both climate subsystems: for instance, the way in which averaging processes in media with different characteristic scales may produce the presence of memory terms in the averaged equations can be found in many texts (see, for instance, [59]). This explains why different delayed terms, or more generally speaking non local terms, may arise in the modelling of the Energy Balance Models due to the own averaging method (see, e.g. the exposition made in [45] and the mathematical treatment made in [31] and [32]. But, some times, the presence of some memory terms can be argued from other modelling proposals (for instance, [6] study the effects of a delayed term to take into consideration the important role of the clouds on the albedo). Moreover, there are many reasons to consider the occurrence of a general delay term in some of the differential equations. For instance, a different justification can be obtained by regarding some other key phenomena, such as the El Niño/Southern Oscillation (ENSO) in the Tropical Pacific and its implications in the climate's interannual variability and in global warming. Although in this case the time scale must be shorter than in other Paleoclimate models, we recall that in many previous EBMs seasonal effects have been taken into account (for instance, the insolation function  $S(t, x)$  is then taken as time-dependent) and so, some justification of the

past glaciations where obtained in [54]. Here we shall only include a delayed term in the deep ocean temperature equation following the approach initiated in the papers [14], [63], [21], [46], etc., to simulate seasonally-varying internal parameters.

As said before, even if the model under consideration responds to simplified modelling arguments, the presence of several nonlinear terms, some of them not always differentiable, makes that its mathematical treatment can not be reduced to the mere direct application of the differential delayed equations (DDE) theory (see, e.g. [47] and [71]). In Section 2 we state the model under consideration and the main structural and constitutive assumptions. The study of solutions of the transient regime is presented in Section 3. Since there is no hope to get classical solutions of the system, we introduce the notion of weak solution we shall deal with. We prove the existence of such type of solutions under quite general assumptions on the data and, which is more unusual in the study of parabolic type systems, we prove that, in general, there is no uniqueness of solutions when the coalbedo is assumed to be discontinuous. Since we also prove that, in this case, there is a continuum of solutions for suitable initial data, it is not possible to apply the results of the classical bifurcation theory for transient systems. Instead of that, we prove the uniqueness of weak non-degenerate solutions (corresponding to the case in which the atmospheric surface temperature arrives not too flat near the boundary of the region where the coalbedo changes, i.e. on the surface where it becomes abruptly discontinuous). Let us recall that new aspects which have been taken into consideration in this type of coupled models are the ocean latent heat and the presence of a memory term.

Once we know the global existence of solutions on any arbitrary time horizon  $T$  it can be proved (see [30] and [64] for a special case of the present system) that the assumptions made here on the data exclude any other elements in the omega limit set (when  $t \rightarrow +\infty$ ) different than the solutions of the stationary system. Perhaps this is the moment to point out that in many other systems the memory may lead to different qualitative properties of solutions with respect to the same system but without memory. We shall not develop this approach here but we refer the reader to a series of papers where this philosophy was carried out for different types of delayed systems (see [15], [16] and [17]).

Coming back to the consideration of the associated stationary system, we show, in Section 4, the multiplicity of solutions in terms of the solar constant  $Q$ . Again, our result is not an automatic application of the general bifurcation theory but requires the construction of suitable families of super and subsolutions well adapted to our setting. An S-shaped bifurcation curve can be obtained in some special cases (see [2]).

The above mentioned mathematical analysis of the model allows to start the study of the controllability of some models connected with the climate system and, in particular with EBMs and related problems (see [25] and [26] for the case of a single EBM equation and [34] and [20] for some related problems). Moreover, it is possible to get a mathematical meaning to the proposals already present in some late works by J. von Neumann (see [68] and [29]).

Finally, in Section 5, we present several numerical experiments on the coupled model by means of a finite volume approach with WENO-7 spatial reconstruction and third order TVD Runge-Kutta for time discretization (for the application of the finite element method see [9]). We compare the numerical solution of the model with and without the effect of the ocean latent heat, and we also present a numerical experiment carried out by considering the effects of the memory term. Although the data in such experiments could be made more realistic we think that the main value of such numerical approach is to show how it is possible to made accessible to the quantification some sophisticated mathematical analysis of complex nonlinear systems, involving, for instance discontinuous albedo data, for which the solutions satisfy the requirements only in a weak sense. As Jacques-Louis Lions (1928 – 2001) said: *if we accept that a model without data is a worthless predictive model, it is also true that data without a good model produce only confusion* (quoted in [57]).

## 2. The mathematical model

Energy Balance Models (EBMs) were introduced, independently by M. I. Budyko [1969] and W. D. Sellers [1969] (some pioneering model is due to S. Arrhenius in 1896). Such type of climatological models have a diagnostic character and intended to understand the evolution of the global climate on a long time scale (see, e.g. [19], [48], [45]). Their main characteristic is the high sensitivity to the variation of solar and terrestrial parameters. They have been used in the study of the *Milankovitch theory* of the ice-ages (see, e.g. [54]).

The EBMs study a distribution of surface atmospheric temperature,  $u(t, x)$ , which is expressed pointwise after some averaging process in space (the spatial variable  $x$  is in a small neighborhood  $B_\delta(x)$  in the Earth's surface) and in time (on a small interval  $(t - \tilde{t}, t + \tilde{t})$ )

$$u(t, x) = \frac{1}{2\tilde{t}|B_\delta(x)|} \int_{t-\tilde{t}}^{t+\tilde{t}} \int_{B_\delta(x)} T(a, s) da ds.$$

The pointwise temperature  $T(a, s)$  is obtained from the thermodynamics equation of the *atmosphere primitive equations* (see e.g., [53] for a mathematical study of those equations and [52], [31] for the application of averaging processes in this context).

More simply, the energy balance model can be formulated by using the energy balance on the Earth's surface: internal energy flux variation =  $R_a - R_e + D$ , where  $R_a$  (respectively  $R_e$ ) represents the absorbed solar (resp. the emitted terrestrial energy flux) and where  $D$  is the surface heat diffusion. By identifying the Earth's surface with a compact Riemannian manifold without boundary  $\mathcal{M}$  (for instance, the sphere  $S^2$  in  $\mathbb{R}^3$ ), the distribution of temperature,  $u(t, x)$ , becomes a function of the spatial  $x$  and  $t$  time variables. The time scale is considered relatively long. The *absorbed energy*  $R_a$  depends on the planetary *coalbedo*  $\beta$ . The coalbedo function represents the fraction of the incoming radiation flux which is absorbed by the surface. In ice-covered zones, reflection is greater than over oceans, therefore, the coalbedo is smaller. One observes that there is a sharp transition between zones of high and low coalbedo. In the energy balance climate models, a main change of the coalbedo occurs in a neighborhood of a critical temperature for which ice become white, usually taken as  $u = -10^0 C$ . The coalbedo can be modelled by different monotone increasing fuctions (discontinuous in case of *Budyko model* and Lipschitz continuous for *Sellers model*). A more realistic albedo parametrization can be obtained by assuming that the coalbedo fuction  $\beta$  also depends on the spatial coordinates of each point of the Earth (specially on its latitude: see [48], Section 3.3). Here we mainly consider the Budyko model since it produces more clear answers when one studies the evolution of the ice caps.

With respect to the surface temperature diffusion we send the reader to the modeling performed for instance in [45] for the case of a linear second order differential operator. Nevertheless, a quasilinear diffusion operator of the type  $\text{div}(k(x, u, \nabla u) \nabla u)$  was proposed in Stone [1971] as a better eddy diffusive approximation to account for the effect of large scale atmospheric circulation, where  $k(x, u, \nabla u)$  is a non linear eddy diffusion coefficient, in particular,  $k = b(x) |\nabla u|$ . In our model, we shall follow Stone's approach to represent the eddy diffusive terms by setting  $k(x, u, \nabla u) = k(x) |\nabla u|^{p-2}$ , with  $p \geq 2$  and  $k(x) > \alpha > 0$ .

With respect to the simplified model on the deep ocean we shall follows the modeling derived in [70] but adding a positive latent heat,  $\gamma$ , which plays an important role in the formation of ice sheets. With respect to the memory term we recall that such type of terms were proposed for the study of ENSO events. For instance, in [63] it is taken  $\mathcal{G}(t, x, u, u(t - \tau)) = -u + u^3 + \alpha u(t - \tau)$ , for some  $\alpha, \tau > 0$ . We could include also some memory terms inside the albedo and latent heat expressions (as in [58]) but the detailed mathematical treatment is much more technical. Notice that since  $u$  will be a globally bounded function, without lost of generality, we can modify the previous example function outside a compact of  $\mathbb{R}^2$  (concerning the values of  $u$  and  $u(t - \tau)$ ) in order to get a globally Lipschitz function. Obviously, the case  $\mathcal{G}(t, x, u, u(t - \tau)) = \mathcal{G}(t, x, u)$  represents the case without delayed effects, such as it was considered by many previous authors. Notice also, that if  $\tau > 0$  then the initial condition for the unknown  $u$  needs to be given on the set  $[-\tau, 0] \times \mathcal{M}$ .

Summarizing, our model will represent the interior and surface temperature of a global ocean  $\Omega$ , so that, the unknown are respectively given by  $U : \Omega \times [0, T] \rightarrow \mathbb{R}$  and by  $u : \mathcal{M} \times [-\tau, T] \rightarrow \mathbb{R}$ , for an arbitrary  $T > 0$ . Here we assume

- (H1)  $\Omega$  is a bounded and open set of  $\mathbb{R}^3$  with maximum depth  $H$  and  $\partial\Omega = \mathcal{M} \cup \mathcal{N}$ .  $\mathcal{M}$  and  $\mathcal{N}$  are  $C^\infty$  two dimensional compact connected oriented Riemannian Manifold of  $\mathbb{R}^3$  without boundary and  $\text{dist}(\mathcal{M}, \mathcal{N}) = H$ .

Let  $(P_{3D})$  be the problem

$$\begin{aligned} \frac{\partial \gamma(U)}{\partial t} - \text{div}(\nabla U) + w \frac{\partial U}{\partial z} &\ni 0 \quad \text{in } (0, T) \times \Omega, \\ \frac{\partial u}{\partial t} - \text{div}(|\nabla_{\mathcal{M}} u|^{p-2} \nabla_{\mathcal{M}} u) + \frac{\partial U}{\partial n} + F(x, \nabla_{\mathcal{M}} u) + \mathcal{G}(t, x, u, u(t-\tau)) &\in \\ &\in \frac{1}{\rho c} Q S(t, x) \beta(u) + f(t, x) \quad \text{in } (0, T) \times \mathcal{M}, \\ U|_{[0, T] \times \mathcal{M}} &= u, \\ \hat{F}(x, \nabla_{\mathcal{N}} U) + \frac{\partial U}{\partial z} &= 0 \quad \text{in } (0, T) \times \mathcal{N}, \\ U(0, x, z) &= U_0(x, z) \quad \text{in } \Omega, \\ u(s, \cdot) &= u_0(s, \cdot) \quad \text{on } [-\tau, 0] \times \mathcal{M}. \end{aligned}$$

Here  $\nabla_{\mathcal{M}}$  and  $\text{div}$  are understood in the sense of the Riemannian metric on  $\mathcal{M}$  (see, e.g. [38] and [53]). The rest of structural conditions are the following:

- (H2)  $\beta$  is a bounded maximal monotone graph, i.e.  $|v| \leq M$  for all  $v \in \beta(s)$ , and all  $s \in D(\beta) = \mathbb{R}$ .

- (H3)  $\gamma$  is the graph

$$\gamma(s) = \begin{cases} k_1 s & \text{if } s < 0, \\ [0, L] & \text{if } s = 0, \\ k_2 s + L & \text{if } s > 0, \end{cases}$$

with  $k_1 > 0, k_2 > 0$  and  $L > 0$ .

- (H4)  $\mathcal{G} : (0, T) \times \mathcal{M} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function,  $\mathcal{G}(t, x, \sigma, \eta)$  is a globally Lipschitz function with respect to  $\sigma$  (i.e.  $u$ ) and  $\eta$  (i.e.  $u(t-\tau)$ ), such that  $\mathcal{G}(t, x, 0, 0) = 0$  and  $|\mathcal{G}(t, x, \sigma, \eta)| \geq C(|\sigma|^r + |\eta|^r)$  for some  $r > 0$ . Moreover  $\tau \geq 0$ .

- (H5)  $S : (0, T) \times \mathcal{M} \rightarrow \mathbb{R}$ ,  $S_1 \geq S(t, x) \geq S_0 > 0$  a.e.  $x \in \mathcal{M}$ .

- (H6)  $f \in L^\infty((0, T) \times \Omega)$ .

- (H7)  $F : \mathcal{M} \times T\mathcal{M} \rightarrow \mathbb{R}$  and  $\hat{F} : \mathcal{N} \times T\mathcal{N} \rightarrow \mathbb{R}$  are linear on the tangent bundle spaces  $T\mathcal{M}$  and  $T\mathcal{N}$  with bounded coefficients.

- (H8)  $w \in C^1(\overline{\Omega})$ .

**Remark 2.1.** We point out that, for simplicity, we have assumed here isotropic (and constant) diffusion matrices in both equations. The mathematical treatment of the case of non constant definite positive diffusion matrices is quite similar and we drop the details.

**Remark 2.2.** The case in which the Solar constant  $Q$  is assumed, in fact, as a periodic or almost periodic time function has been intensively studied in the literature (see, e.g. [45], [4] and its many references).

We can reduce the dimension of the model by assuming that the surface  $\mathcal{M}$  is a sphere simulating the Earth surface and that the temperature is constant over each parallel. So, the solution of the obtained model only depends on latitude, depth and time. For different purposes, it is useful to particularize the above system to the simpler case of a 1D EBM (the surface temperature is

defined on  $[0, T] \times [-1, 1]$ : i.e. here  $\mathcal{M} = \{(x, 0) : x \in [-1, 1]\} := \Gamma_0$  coupled with a 2d deep ocean ( $\Omega = [-1, 1] \times [0, -H]$ , and so of boundaries  $\mathcal{N} = \{(x, -H) : x \in [-1, 1]\} := \Gamma_H$ ,  $\Gamma_{-1} := \{(-1, z) : z \in [-H, 0]\}$  and  $\Gamma_1 := \{(1, z) : z \in [-H, 0]\}$ ). The resulting equations of the model (this time with nonisotropic diffusion coefficients and with  $F(x, \nabla_{\mathcal{M}} u) := wxu_x$  and  $\hat{F}(x, \nabla_{\mathcal{N}} U) := wxU_x$  and with a parameter  $D > 0$  modeling the mixed layer depth) now with  $x = \sin \varphi$ ,  $\varphi$  representing the latitude, and  $z \in [-H, 0]$ , are the following:

$$\begin{aligned} \gamma(U)_t - \left(\frac{K_H}{R^2}(1-x^2)U_x\right)_x - K_V U_{zz} + wU_z &\ni 0 \quad \text{in } \Omega \times (0, T), \\ wxU_x + K_V U_z &= 0 \quad \text{in } \Gamma_H \times (0, T), \\ Du_t - \frac{DK_{H_0}}{R^2} \left((1-x^2)^{p/2} |u_x|^{p-2} u_x\right)_x + K_V \frac{\partial U}{\partial n} + wxu_x + \mathcal{G}(t, x, u, u(t-\tau)) &\in \\ &\in \frac{1}{\rho c} S(t, x) Q\beta(u) + f(t, x) \quad \text{in } \Gamma_0 \times (0, T), \\ U|_{[0, T] \times [-1, 1]} &= u, \\ (1-x^2)U_x &= 0 \quad \text{in } (\Gamma_{-1} \times (0, T)) \cup (\Gamma_1 \times (0, T)), \\ U(0, x, z) &= U_0(x, z) \quad \text{in } \Omega, \\ u(s, x, 0) &= u_0(s, x, 0) \quad \text{on } [-\tau, 0] \times \Gamma_0. \quad (P_{2D}) \end{aligned}$$

**Remark 2.3.** We note that we can introduce the change of variable  $U = \alpha(V)$ , with  $\alpha := \gamma^{-1}$ , and then the equation in the inner ocean can be written as

$$V_t - \left(\frac{K_H}{R^2}(1-x^2)\alpha(V)_x\right)_x - K_V \alpha(V)_{zz} + w\alpha(V)_z = 0 \quad \text{in } \Omega \times (0, T),$$

where

$$\alpha(s) = \begin{cases} \frac{s}{k_1} & \text{if } s < 0, \\ 0 & \text{if } 0 < s < L, \\ \frac{1}{k_2}(s-L) & \text{if } s > L. \end{cases} \quad (2.1)$$

The terms  $\gamma$  and  $\alpha$  (as well as  $\beta$ ) are maximal monotone graphs (see [11]). The main difference between  $\gamma$  and  $\alpha$  is that  $\gamma$  is always multivalued (once we assume  $L > 0$ ) although, in the atmosphere temperature equation, the coalbedo  $\beta$  becomes a multivalued graph only when it is associated to a discontinuous coalbedo function, such as it was proposed in [13]. This is the reason why in the previous inner ocean equation and the surface EBM it appears the symbols  $\in$  and  $\ni$  instead of the usual equality symbol.

### 3. On the evolution problem

#### 3.1 Existence of solutions.

We define the functional space  $V := \{u \in L^2(\mathcal{M}) : \nabla_{\mathcal{M}} u \in L^p(T\mathcal{M})\}$ , where  $T\mathcal{M} = \cup_{p \in \mathcal{M}} T_p \mathcal{M}$  is the tangent bundle space (see [3]). Due to the presence of possible multivalued graphs (associated to discontinuous functions), and the possible choice  $p \neq 2$ , we can not expect to solve the system in a classical sense but only in a weak way.

We say that the pair  $(U, u)$  with  $U \in C([0, T] : L^2(\Omega))$ ,  $u \in C([-\tau, T] : L^2(\mathcal{M}))$  is a bounded weak solution of  $(P_{3D})$  if

- (i)  $(U, u) \in L^\infty((0, T) \times \Omega) \times L^\infty((-\tau, T) \times \mathcal{M}) \cap L^2(0, T : H^1(\Omega)) \times L^p(-\tau, T : V)$ ,
- (ii) there exist  $Z \in L^\infty((0, T) \times \Omega)$  and  $h \in L^\infty((-\tau, T) \times \mathcal{M})$  with  $Z \in \gamma(U)$  a.e.  $(t, x) \in (0, T) \times \mathcal{M}$ ,  $h \in \beta(u)$  a.e.  $(t, x) \in (-\tau, T) \times \mathcal{M}$  and such that

$$\begin{aligned}
& \int_{\Omega} Z(T, x) \phi(T, x) dA - \int_0^T \langle \phi_t(t, x), Z(t, x) \rangle_{H^1(\Omega) \times H^1(\Omega)'} dt + \int_0^T \int_{\Omega} \nabla U \nabla \phi dA dt + \\
& + \int_0^T \int_{\Omega} w \frac{\partial U}{\partial z} \phi dA dt - \int_0^T \int_{\mathcal{M}} \frac{\partial U}{\partial n} \phi dS dt + \int_0^T \int_{\mathcal{N}} \hat{F}(x, \nabla_{\mathcal{N}}) \phi dS dt \\
& = \int_{\Omega} U_0(x) \phi(0, x) dA,
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\mathcal{M}} u(T, x) \psi(T, x) dA - \int_0^T \langle \psi_t(t, x), u(t, x) \rangle_{V' \times V} dt + \int_0^T \int_{\mathcal{M}} |\nabla u|^{p-2} \nabla u \nabla \psi dS dt + \\
& + \int_0^T \int_{\mathcal{M}} \mathcal{G}(t, x, u, u(t - \tau)) \psi dS dt + \int_0^T \int_{\mathcal{M}} \frac{\partial U}{\partial n} \psi dS dt + \int_0^T \int_{\mathcal{M}} F(x, \nabla_{\mathcal{M}}) \psi dS dt \\
& = \int_0^T \int_{\mathcal{M}} Q S(t, x) h(t, x) \psi dA dt + \int_0^T \int_{\mathcal{M}} f \psi dA dt + \int_{\mathcal{M}} u_0(0, x) \psi(0, x) dS
\end{aligned}$$

for every test function  $(\phi, \psi) \in L^2(0, T; H^1(\Omega)) \times L^p(-\tau, T; W^{1,p}(\mathcal{M}))$  such that  $(\phi_t, \psi_t) \in L^2(0, T; H^1(\Omega)') \times L^{p'}(0, T; V')$ . Here  $\langle \cdot, \cdot \rangle_{V' \times V}$  denotes the duality product in  $V' \times V$ .

**Theorem 3.1.** *Let  $U_0 \in L^\infty(\Omega)$  and  $u_0 \in C((-\tau, 0]; L^\infty(\mathcal{M}))$ . Then there exists at least a bounded weak global solution of  $(P_{3D})$ .*

*Proof.* We write the inner ocean equation as

$$\frac{\partial V}{\partial t} - \operatorname{div}(\nabla \alpha(V)) + w \frac{\partial \alpha(V)}{\partial z} = 0 \quad \text{in } (0, T) \times \Omega,$$

with  $U = \alpha(V)$  and  $\alpha := \gamma^{-1}$ , as mentioned in the above Remark 2.3 (notice that now  $\alpha$  is singlevalued and so we do not need the symbol  $\epsilon$ ). We approximate the maximal monotone graph  $\alpha$  by some smooth increasing functions  $\alpha_\epsilon$ . Then we obtain a family of new problems, that we shall denote by  $(P_\epsilon)$ . The main idea to solve  $(P_\epsilon)$  is to apply Theorem 5.3.1 of [69] related to abstract functional equations. We shall construct an operator  $\mathcal{T}_\epsilon$  and to find a fixed point of it leading to a solution of  $(P_\epsilon)$ . This will consist of several intermediate steps.

**Step 1.** For every  $h \in L^\infty((0, T) \times \mathcal{M})$  we consider the problem  $(P_{h, \epsilon})$  by replacing the coalbedo term in  $(P_\epsilon)$  by  $h$ . The proof of the existence of solution of  $(P_{h, \epsilon})$  is inspired in [35] and [7]. We define the vectorial operator  $\mathcal{A}_\epsilon$  by  $\mathcal{A}_\epsilon(U, u) \mapsto (A_\epsilon U, Bu)$  on the domain  $D(\mathcal{A}_\epsilon) = \{(U, u) \in L^2(\Omega) \times L^2(\mathcal{M}) : A_\epsilon U \in L^2(\Omega), Bu \in L^2(\mathcal{M}), \alpha_\epsilon(U)|_{\mathcal{M}} = u\}$ , where

$$\begin{aligned}
A_\epsilon U &= -\operatorname{div}(\nabla \alpha_\epsilon(U)) + w \frac{\partial \alpha_\epsilon(U)}{\partial z}, \\
Bu &= -\operatorname{div}(|\nabla_{\mathcal{M}} u|^{p-2} \nabla_{\mathcal{M}} u) + \frac{\partial \alpha_\epsilon(U)}{\partial n} + F(x, \nabla_{\mathcal{M}} u).
\end{aligned}$$

We also define the operator  $G(t)u := \mathcal{G}(t, x, u, u(t - \tau))$ . Then, the existence of solution of  $(P_{h, \epsilon})$  is a consequence of the compactness of the semigroup associated to the operator  $\mathcal{A}_\epsilon(U, u)$  (through Theorem 5.3.1 of [69]) and the results of [35] and [7] leading, up very small variations, to the following properties of  $\mathcal{A}_\epsilon$ .

**Lemma 3.1.** *There exists  $\lambda_0 > 0$  such that for every  $\lambda > \lambda_0$ , we have:*

- (i)  $\mathcal{A}_\epsilon + \lambda I$  is  $T$ -accretive in  $L^1(\Omega) \times L^2(\mathcal{M})$ .
- (ii)  $R(\mathcal{A}_\epsilon + \lambda I) = L^1(\Omega) \times L^2(\mathcal{M})$ .

Note that (i) allows us to prove a comparison principle for the system

$$\begin{aligned}\lambda U + A_\epsilon U &= f \quad \text{in } L^1(\Omega), \\ \lambda u + B u &= g \quad \text{in } L^2(\mathcal{M}), \\ \alpha_\epsilon(U)|_{\mathcal{M}} &= u, \\ \widehat{F}(x, \nabla_{\mathcal{N}} \alpha_\epsilon(U)) + \frac{\partial \alpha_\epsilon(U)}{\partial z} &= 0 \quad \mathcal{N}.\end{aligned}\tag{3.1}$$

In fact, if  $f_1 \leq f_2$  and  $g_1 \leq g_2$  then the solutions of (3.1) with  $f = f_1, g = g_1$  and of (3.1) with  $f = f_2, g = g_2$  satisfy  $U_1 \leq U_2$  and  $u_1 \leq u_2$ .

The small variation with respect to the proof given in [7] concerns the proof of (ii) in Lemma 3.1. We notice that the operator  $B$  can be expressed as  $B_1 + B_2 + B_3$ , where  $B_1$  and  $B_2$  are maximal monotone operators in  $L^2(\mathcal{M})$ ,

$$B_1 u = -\operatorname{div}(|\nabla_{\mathcal{M}} u|^{p-2} \nabla_{\mathcal{M}} u)$$

and the pseudo-differential operator  $B_2 u = \frac{\partial \alpha_\epsilon(U)}{\partial n}$ , where  $U$  is the solution of the problem

$$\begin{aligned}\lambda U + A_\epsilon U &= f \quad \text{in } L^2(\Omega) \\ \alpha_\epsilon(U)|_{\mathcal{M}} &= u.\end{aligned}$$

The operator  $B_3$  is defined by  $B_3 u = F(\nabla_{\mathcal{M}} u)$ . This operator is not necessarily monotone but it is dominated (in the usual sense: see [11]) by the operators  $B_1$  and  $B_2$ . Consequently, it is possible to apply the abstract results of perturbation of maximal monotone operators (see e.g. Proposition 2.10 of [11]) and we arrive to the desired conclusion.

**Step 2.** We follow closely the proof of Theorem 5.3.1 of [69] and the one given in Theorem 3 of [38] for a related problem. We define the operator  $\mathcal{T}_\epsilon : h \rightarrow g$  where  $g \in \beta(u_h)$  and  $u_h$  is the solution of  $(P_h)$ . It is easy to see that every fixed point of  $\mathcal{T}_\epsilon$  is a solution of  $(P_\epsilon)$ . Moreover,  $\mathcal{T}_\epsilon$  satisfies the hypotheses of Kakutani fixed point Theorem (see e.g. [69]), and so, if we denote  $X = L^p(0, T : L^2(\mathcal{M}))$ , then

- (i)  $K = \{h \in L^p((0, T), L^\infty(\Omega)) : \|h(t)\| \leq C_0 \text{ a.e. } t \in (0, T)\}$  is a nonempty, convex and weakly compact set of  $X$ ;
- (ii)  $\mathcal{T}_\epsilon : K \rightarrow 2^X$  with nonempty, convex and closed values such that  $\mathcal{T}_\epsilon(g) \subset K, \forall g \in K$ ;
- (iii)  $\operatorname{graph}(\mathcal{T}_\epsilon)$  is weakly  $\times$  weakly sequentially closed.

Consequently,  $\mathcal{T}_\epsilon$  has at least one fixed point in  $K$ . Finally, arguing as in the proof of Theorem 5.3.1 of [69] we prove the existence of a weak solution of  $(P_\epsilon)$ .

Finally, we shall pass to the limit when  $\epsilon \rightarrow 0$ . To do that we shall use several a priori estimates. Firstly, due to the assumptions on the initial data and the above Lemma 3.2 we know that there exists  $M > 0$ , independent of  $\epsilon$ , such that

$$\max(\|U_\epsilon\|_{L^\infty((0, T) \times \Omega)}, \|u_\epsilon\|_{L^\infty((-\tau, T) \times \mathcal{M})}) \leq M$$

and (by multiplying by  $U_\epsilon$  and  $u_\epsilon$  in the respective equations)

$$\max(\|U_\epsilon\|_{L^2(0, T; H^1(\Omega))}, \|u_\epsilon\|_{L^p(-\tau, T; V)}) \leq M.$$

We also have that  $u_\epsilon$  is a strong solution (see [38]) in the sense that

$$\left\| \frac{\partial u_\epsilon}{\partial t} \right\|_{L^2((-\tau, T) \times \mathcal{M})} \leq M,$$

and that the family  $\{U_\epsilon\}$  is equicontinuous (see Proposition 6.3 of [22]). Then there exists a subsequence of  $\{U_\epsilon\}$  and  $\{u_\epsilon\}$  (which we still label in the same way) such that  $U_\epsilon \rightarrow \hat{U}$  weakly in  $L^2(0, T : H^1(\Omega))$  and strongly in  $C([0, T] : L^1(\Omega))$  (respectively  $u_\epsilon \rightarrow u$  weakly in  $L^p(-\tau, T : V)$  and strongly in  $C([-\tau, T] : L^1(\Omega))$ ). Finally, by using that  $\gamma$  and  $\beta$  are maximal monotone graphs,



and assumption (H<sub>4</sub>) on  $\mathcal{G}(t, x, \sigma, \eta)$ , we can pass to the limit in all terms and we conclude that  $(U, u)$ , where  $U = \alpha(\hat{U})$ , is a weak solution of the original problem  $(P_{3D})$ .

**Remark 3.1.** Lemma 3.1 and similar arguments to those in Lemma 3 of [38] allow us to prove the existence of a maximal and minimal solutions.

### 3.2 Nonuniqueness of solutions in the presence of a discontinuous coalbedo term.

The presence of the multivalued coalbedo,  $\beta$ , (corresponding to a discontinuous function which graph is completed as to generate a maximal monotone graph) allows us to prove that, for some special initial data, there exist more than one time dependent solution. We assume here the following conditions.

(H<sub>1</sub><sup>\*</sup>) The coalbedo function is

$$\beta(u) = \begin{cases} [m, M] & \text{if } u = -10, \\ m & \text{if } u < -10, \\ M & \text{if } u > -10, \text{ with } 0 < m < M. \end{cases} \quad (3.2)$$

(H<sub>2</sub><sup>\*</sup>)  $\mathcal{G}(t, x, u, u(t - \tau)) = Bu + C - \mu u(t - \tau)$  and  $\gamma(u) = u$ .

(H<sub>3</sub><sup>\*</sup>)  $B$  and  $C$  are positive constants verifying

$$\frac{Qs_1m}{\rho c} < -10B + C, \quad -10B + C + \mu \|u_0\|_{L^\infty(-\tau, 0) \times L^\infty(-1, 1)} < \frac{Qs_0M}{\rho c}. \quad (3.3)$$

(H<sub>4</sub><sup>\*</sup>) We also assume  $w(x) \leq 0$  for all  $x \in (-1, 1)$ .

(H<sub>5</sub><sup>\*</sup>) The initial data  $(U_0, u_0)$  satisfy

$$\left\{ \begin{array}{l} U_0 \in C^\infty(\Omega), \quad u_0 \in C([-\tau, 0]) \times C^\infty(\Gamma_0), \quad u_0(s, x) = u_0(s, -x) = u_0(0, x), \quad x \in [-1, 1], \quad s \in [-\tau, 0] \\ \\ \frac{\partial u_0}{\partial x}(s, 0) = \frac{\partial^2 u_0}{\partial x^2}(s, 0) = 0, \quad u_0(s, 0) = -10, \\ \\ \frac{\partial u_0}{\partial x}(s, x) < 0 \text{ if } x \in (0, 1), \quad \frac{\partial u_0}{\partial x}(s, 1) = 0, \quad s \in (-\tau, 0] \\ \\ \frac{\partial U_0}{\partial z}(x, 0) > 0, \quad U_0(x, 0) = u_0(0, x), \text{ if } x \in (0, 1). \end{array} \right.$$

**Theorem 3.2.** Under the above conditions, Problem  $(P_{2D})$  has at least two bounded weak solutions.

*Proof.* Step 1. First, we consider the problem  $(P_m)$

$$\left\{ \begin{array}{ll} \frac{\partial U}{\partial t} - \frac{K_H}{R^2} \frac{\partial}{\partial x} ((1 - x^2) \frac{\partial U}{\partial x}) - K_v \frac{\partial^2 U}{\partial z^2} + w \frac{\partial U}{\partial z} &= 0 \quad (0, T) \times \Omega, \\ \\ wx \frac{\partial U}{\partial x} + K_V \frac{\partial U}{\partial z} &= 0 \quad (0, T) \times \Gamma_H \\ \\ D \frac{\partial U}{\partial t} - \frac{DK_{H_0}}{R^2} \frac{\partial}{\partial x} \left( (1 - x^2)^{\frac{p}{2}} \left| \frac{\partial U}{\partial x} \right|^{p-2} \frac{\partial U}{\partial x} \right) + \\ + K_V \frac{\partial U}{\partial n} + wx \frac{\partial U}{\partial x} + Bu + C - \mu u(t - \tau, x) &= \frac{1}{\rho c} Q S(x) m \quad \text{on } (0, T) \times \Gamma_0 \\ \\ (1 - x^2)^{\frac{p}{2}} \left| \frac{\partial U}{\partial x} \right|^{p-2} \frac{\partial U}{\partial x} &= 0 \quad \Gamma_1 \cup \Gamma_{-1} \\ U(0, x, z) = U_0(x, z) &\quad \Omega, \\ U(0, x, 0) = u_0(x) &\quad (-1, 1), \end{array} \right.$$

We notice that if  $t < \tau$  then  $u(t - \tau, x) = u_0(t - \tau)$ . Denote  $(U^m, u^m)$  to the solution of  $(P_m)$ . We notice that if  $u^m \leq -10$  then  $(U^m, u^m)$  is also a solution of  $(P_{2D})$  because  $h(t, x) \equiv m \in \beta(u_m)$ .

Now, by changing  $U^* = -10 - U^m$  and  $u^* = -10 - u^m$ , we have that  $u^*$  verifies

$$Du_t^* - \frac{DK_{H_0}}{R^2}((1-x^2)|u_x^*|^{p-2}u_x^*)_x + Bu^* = -\frac{QSm}{\rho c} - 10B + C - K_V \frac{\partial U^*}{\partial n} - wx \frac{\partial u^*}{\partial x} - \mu u_0(t-\tau, x).$$

From hypotheses  $(H_3^*)$  and  $(H_5^*)$ , there exists  $T_0 > 0$  s.t. if  $t < T_0$  then the right hand side term is positive. Consequently  $u^* = -10 - u^m$  is positive and  $u^m < -10$ . Notice that  $K_V \frac{\partial U}{\partial n} + wx \frac{\partial U}{\partial x} \leq 0$  in  $(0, T_0) \times \Gamma_0$ .

Step 2. Now, we prove that there exist a solution which takes values bigger than -10 in a subset of  $\Gamma_0$  for  $t < \tau$ . To see the existence of this second solution, we shall construct a family of auxiliary functions  $U^\lambda$  (and the restrictions  $U|_{\Gamma_0} = u^\lambda$ ). We decompose  $\Omega \times [0, \lambda] = \mathcal{Q}_1^\lambda \cup \mathcal{Q}_2^\lambda \cup \Sigma^\lambda$ , where

$$\mathcal{Q}_1^\lambda = \{(x, z, t) \in \Omega \times [0, \lambda] : x^2 + z^2 > \frac{t^2}{\lambda^2}\},$$

$$\mathcal{Q}_2^\lambda = \{(x, z, t) \in \Omega \times [0, \lambda] : x^2 + z^2 < \frac{t^2}{\lambda^2}\},$$

$$\Sigma^\lambda = \{(x, z, t) \in \Omega \times [0, \lambda] : x^2 + z^2 = \frac{t^2}{\lambda^2}\}.$$

In the region  $\mathcal{Q}_1^\lambda$ . We consider  $(U^\lambda, u^\lambda)$  the solution of problem  $(P_{\mathcal{Q}_1^\lambda})$  (see e.g. [44] and [41]).

$$\left\{ \begin{array}{ll} \frac{\partial U}{\partial t} - \frac{K_H}{R^2} \frac{\partial}{\partial x}((1-x^2) \frac{\partial U}{\partial x}) - K_v \frac{\partial^2 U}{\partial z^2} + w \frac{\partial U}{\partial z} = 0 & \mathcal{Q}_1^\lambda \\ wx \frac{\partial U}{\partial x} + K_V \frac{\partial U}{\partial z} = 0 & \mathcal{Q}_1^\lambda \cap (0, T) \times \Gamma_H \\ D \frac{\partial U}{\partial t} - \frac{DK_{H_0}}{R^2} \frac{\partial}{\partial x} \left( (1-x^2)^{\frac{p}{2}} \left| \frac{\partial U}{\partial x} \right|^{p-2} \frac{\partial U}{\partial x} \right) + \\ + K_V \frac{\partial U}{\partial n} + wx \frac{\partial U}{\partial x} + Bu + C - \mu u_0 = \frac{1}{\rho c} QS(x)m & \mathcal{Q}_1^\lambda \cap (0, T) \times \Gamma_0 \\ (1-x^2)^{\frac{p}{2}} \left| \frac{\partial U}{\partial x} \right|^{p-2} \frac{\partial U}{\partial x} = 0 & \text{on } \Gamma_1 \cup \Gamma_{-1} \\ U(0, x, z) = U_0(x, z), U(0, x, 0) = u_0(x) & \\ U^\lambda = -10 & \Sigma^\lambda \end{array} \right.$$

On the region  $\mathcal{Q}_2^\lambda$ , we define  $U^\lambda = -10 - C^\lambda(t)(x^2 + z^2 - \frac{t^2}{\lambda^2})$ . Notice that if  $C^\lambda > 0$  then  $U^\lambda > -10$  in  $\mathcal{Q}_2^\lambda$ . Is easy to see that  $(U^\lambda, u^\lambda)$  is a solution of Problem  $(P_\lambda)$ ,

$$\left\{ \begin{array}{ll} \frac{\partial U}{\partial t} - \frac{K_H}{R^2} \frac{\partial}{\partial x}((1-x^2) \frac{\partial U}{\partial x}) - K_v \frac{\partial^2 U}{\partial z^2} + w \frac{\partial U}{\partial z} = H^\lambda & \text{in } (0, T) \times \Omega, \\ wx \frac{\partial U}{\partial x} + K_V \frac{\partial U}{\partial z} = g^\lambda & \text{in } (0, T) \times \Gamma_H \\ D \frac{\partial U}{\partial t} - \frac{DK_{H_0}}{R^2} \frac{\partial}{\partial x} \left( (1-x^2)^{\frac{p}{2}} \left| \frac{\partial U}{\partial x} \right|^{p-2} \frac{\partial U}{\partial x} \right) + \\ + K_V \frac{\partial U}{\partial n} + wx \frac{\partial U}{\partial x} + Bu + C - \mu u_0 = h^\lambda & \text{in } (0, T) \times \Gamma_0 \\ (1-x^2)^{\frac{p}{2}} \left| \frac{\partial U}{\partial x} \right|^{p-2} \frac{\partial U}{\partial x} = 0 & \text{in } (0, T) \times (\Gamma_1 \cup \Gamma_{-1}) \\ U(0, x, z) = U_0(x, z) \text{ in } \Omega, & U(0, x, 0) = u_0(x) \text{ in } (-1, 1), \end{array} \right.$$

where, for  $(t, x, z) \in \mathcal{Q}_2^\lambda$ ,

$$\begin{aligned}
H^\lambda &= -(C^\lambda)'(t)(x^2 + z^2 - \frac{t^2}{\lambda^2}) - C^\lambda(t)[(-\frac{2t}{\lambda^2}) - \frac{2K_H}{R^2}(1 - 3x^2) - 2K_v + 2wz], \\
h^\lambda &= -D(C^\lambda)'(t)(x^2 - \frac{t^2}{\lambda^2}) - C^\lambda(t)[- \frac{2Dt}{\lambda^2} + 2wx^2 + B(x^2 - \frac{t^2}{\lambda^2}) \\
&\quad - 2^{p-1} \frac{DK_{H_0}}{R^2} |C^\lambda(t)|^{p-2} (-p(1-x^2)^{\frac{p-2}{2}} |x|^p + \\
&\quad + (p-1)(1-x^2)^{\frac{p}{2}} |x|^{p-2}) - 10B + C - \mu u_0), \\
g^\lambda &= -2C^\lambda(t)(x^2 w - K_v H) \geq 0.
\end{aligned}$$

Thus, there exist  $\lambda > 0$  and  $C^\lambda : [0, T_0] \rightarrow \mathbb{R}$  such that  $h^\lambda \leq \frac{Qs_0 M}{\rho c}$ . Then  $(U^\lambda, u^\lambda)$  is a lower solution of Problem  $(P_{2D})$ . Then, by upper and lower solution method we deduce that there exists a solution  $(V, v)$  of  $(P_{2D})$  satisfying  $u^\lambda < v$ . Consequently  $v > -10$  in some subset of positive measure.  $(V, v)$  is different than the solution of step 1. Finally, we get two different solutions of  $(P_{2D})$  for an initial data satisfying  $(H_5^*)$ .

**Remark 3.2.** The above construction makes arise a parameter  $\lambda$  which is not uniquely determined. So, in fact, the proof shows the existence of a continuum of solutions, and not only two of them.

**Remark 3.3.** In the proof of the above result, the multivalued nature of  $\beta$  was a crucial element. As a matter of fact, if by the contrary we assume that  $\beta$  is a regular function, for instance a Lipschitz function then, by standard arguments we get the uniqueness of weak solutions.

### 3.3 Uniqueness of non degenerate solutions.

Now, we wonder if it is possible to get uniqueness of time dependent solutions for a model which may involve a multivalued coalbedo term but for some special initial data. The answer is positive but it will depend on a suitable property which must be satisfied by the weak solutions. By simplicity in the exposition we shall assume here  $\gamma(s) = s$  (the result remains true for the case of the graph  $\gamma$  corresponding to a positive latent heat but the details are too technical as to be presented here). We define a class of solutions called as *non degenerate* on  $\Gamma_0$ . This notion was also useful in [24] and [38] where the EBM model without the deep ocean effect was studied.

**Definition.** Let  $w \in L^\infty(\Gamma_0)$ . We say that  $w$  satisfies the strong nondegeneracy property (resp. weak) if there exist  $C > 0$  and  $\epsilon_0 > 0$  such that for any  $\epsilon \in (0, \epsilon_0)$ ,  $|\{x \in \Gamma_0 : |w(x) + 10| \leq \epsilon\}| \leq C\epsilon^{p-1}$  (resp.  $|\{x \in \Gamma_0 : 0 < |w(x) + 10| \leq \epsilon\}| \leq C\epsilon^{p-1}$ ).

**Theorem 3.3.** (i) Assume that there exists a solution  $(U, u)$  of  $(P_{2D})$  such that  $u(t)$  verifies the strong nondegeneracy property for all  $t \in [0, T]$  then  $(U, u)$  is the unique bounded weak solution of  $(P_{2D})$ . (ii) There exists at most one solution of  $(P_{2D})$  verifying the weak nondegeneracy property.

The idea of the proof is based on the fact that  $\beta$  generates a continuous operator from  $L^\infty(\Gamma_0)$  to  $L^q(\Gamma_0) \forall q \in [1, \infty)$  when the domain of such operator is the set of functions verifying the strong nondegeneracy property. More precisely, we estimate the difference between two possible solutions  $(U - V, u - v)$  by using the following

**Lemma 3.2.** (i) Let  $w, \hat{w} \in L^\infty(\Gamma_0)$ . Assume  $w$  satisfies the strong nondegeneracy property. Then, for every  $q \in [1, \infty)$  there exists  $\tilde{C} > 0$  such that for every  $z, \hat{z} \in L^\infty(\Gamma_0)$  verifying  $z(x) \in \beta(w(x))$  and  $\hat{z}(x) \in \beta(\hat{w}(x))$  a.e.  $x \in \Gamma_0$ , we have

$$\|z - \hat{z}\|_{L^q(\Gamma_0)} \leq (b_w - b_i) \min\{\tilde{C} \|w - \hat{w}\|_{L^\infty(\Gamma_0)}^{(p-1)/q}, |2|^{1/q}\}. \quad (3.4)$$

(ii) If  $w, \hat{w} \in L^\infty(\Gamma_0)$  satisfy the weak nondegeneracy property then

$$\int_{\Gamma_0} (z(x) - \hat{z}(x))(w(x) - \hat{w}(x)) dA \leq (b_w - b_i) C \|w - \hat{w}\|_{L^\infty(\Gamma_0)}^p. \quad (3.5)$$

The idea of the proof (for the case of the simpler model  $(P_{2D})$ ) of the uniqueness of solution follows closely Theorem 5 of [38]. First we argue on the time interval  $[0, \tau]$  (it is enough to repeat the same arguments on subintervals of length  $\tau$  to get the result on the whole interval  $[-\tau, T]$  for any arbitrary  $T > 0$ ). Assume there exist two solutions  $(U, u)$  and  $(V, v)$ . By using Holder, Young and Friedrich inequalities and the lemma of nondegeneracy property (by introducing a suitable spatial rescaling  $x \mapsto \lambda x$  to estimate some balance of the upper bounds) we obtain that

$$\frac{\partial}{\partial t} \|U - V\|_{L^2(\Omega)}^2 + \frac{\partial}{\partial t} \|u - v\|_{L^2(\Gamma_0)}^2 \leq K_1 \|U - V\|_{L^2(\Omega)}^2 + K_2 \|u - v\|_{L^2(\Gamma_0)}^2.$$

Finally, by Gronwall Lemma, we conclude that  $\|U - V\|_{L^2(\Omega)} = 0$  and  $\|u - v\|_{L^2(\Gamma_0)} = 0$ , which ends the proof.

**Remark 3.4.** *The conclusion of Theorem 3.7 also holds for the  $(P_{3D})$  but its proof becomes more technical. It will be presented in a future work by the authors.*

## 4. Multiplicity of steady states

The analysis of the stabilization, as  $t \rightarrow +\infty$  of the solutions can not be carried out by means of any linearization principle due to the presence of the possible multivalued graphs  $\gamma$  and  $\beta$ . An alternative method consists in to characterize the  $\omega$ -limit set (once it is assumed that  $f(t, \cdot) \rightarrow f_\infty(\cdot)$ , when  $t \rightarrow +\infty$ , in some suitable sense). In that case, it can be shown that, given  $(U, u)$  bounded weak solution of  $(P_{3D})$ , any element of the  $\omega$ -limit set of  $(U, u)$ , defined by  $\omega(U, u) = \{(U_\infty, u_\infty) \in (H^1(\Omega) \times V) \cap L^\infty(\mathcal{M}) \times L^\infty(\mathcal{M}) : \exists t_n \rightarrow +\infty \text{ such that } (U(t_n, \cdot), u(t_n, \cdot)) \rightarrow (U_\infty, u_\infty) \text{ in } L^2(\Omega) \times L^2(\mathcal{M})\}$ , is formed merely by solutions  $(U_\infty, u_\infty)$  of the associate stationary model, which we denote by  $(P_\infty)$ . The proof of this result follows the ideas of [30] (the details will appear in a future work). The associated stationary problem  $(P_\infty)$  consists of the following set of equations:

$$-div(K \nabla U) + w \frac{\partial U}{\partial z} = 0 \quad \text{on } \Omega, \quad (4.1)$$

$$\hat{F}(x, \nabla_{\mathcal{N}} U) + \frac{\partial U}{\partial z} = 0 \quad \text{on } \mathcal{N}, \quad (4.2)$$

$$-div_{\mathcal{M}}(|\nabla_{\mathcal{M}} u|^{p-2} \nabla_{\mathcal{M}} u) + K_V \frac{\partial U}{\partial n} + F(x, \nabla_{\mathcal{M}} u) + \hat{G}(u) \in R_a(u) + f_\infty \text{ on } \mathcal{M}, \quad (4.3)$$

$$U|_{\mathcal{M}} = u, \quad (4.4)$$

where  $\partial\Omega = \mathcal{N} \cup \mathcal{M}$  and with  $\hat{G}(x, u)$  given by the limit of  $\mathcal{G}(t, x, u, u(t - \tau))$  when  $t \rightarrow +\infty$ . In this section, we shall assume the conditions

(H<sub>S</sub>)  $S : \Omega \rightarrow \mathbb{R}$ ,  $S \in L^\infty(-1, 1)$ ,  $S_1 \geq S(x) \geq S_0 > 0$  for some  $S_1 > S_0$ .

(H<sub>G</sub>)  $\hat{G} : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous strictly increasing function such that  $\hat{G}(0) = 0$  and  $\lim_{|s| \rightarrow \infty} |\hat{G}(s)| = +\infty$ .

(H<sub>f</sub>)  $f_\infty \in L^\infty(\Omega)$  and there exist  $C_f > 0$  such that  $-\|f_\infty\|_\infty \leq f_\infty(x) \leq -C_f$  a.e.  $x \in \Omega$ .

(H<sub>β</sub>)  $\beta$  is a bounded maximal monotone graph of  $\mathbb{R}^2$  and there exists two real numbers  $0 < m < M$  and  $\epsilon > 0$  such that  $\beta(r) = \{m\}$  for any  $r \in (-\infty, -10 - \epsilon)$  and  $\beta(r) = \{M\}$  for any  $r \in (-10 + \epsilon, +\infty)$ .

(H<sub>C<sub>f</sub></sub>)  $\hat{G}(-10 - \epsilon) + C_f > 0$  and  $\frac{\hat{G}(-10 + \epsilon) + \|f_\infty\|_\infty}{\hat{G}(-10 - \epsilon) + C_f} \leq \frac{S_0 M}{S_1 m}$ .

(H<sub>w</sub>)  $w \in C^1(\overline{\Omega})$  (for simplicity).

(H<sub>K</sub>) The constants  $K_H, K_V, K_{H_0}, D, R, \rho, c$  and  $Q$  are positive.

One important technique that we shall use in the following result is the continuity of the solutions with respect to the coalbedo  $\beta$ . This allows us the approximation of a discontinuous (i.e. multivalued graph)  $\beta$  by a smoother functions. This also have some implications for the numerical treatment of the model.

**Theorem 4.1.** Let  $(H_S), (H_G), (H_f), (H_w), (H_K)$  and  $(H_\beta)$  be satisfied. Then for any  $Q > 0$  there is a minimal solution  $(\underline{U}, \underline{u})$  (resp. a maximal solution  $(\overline{U}, \overline{u})$ ) of problem  $(P_Q)$ . Moreover, if  $(H_{C_f})$  holds, then there exist  $Q_1 < Q_2 < Q_3 < Q_4$  such that

- i) if  $0 < Q < Q_1$ , then  $(P_Q)$  has a unique solution,
- ii) if  $Q_2 < Q < Q_3$ , then  $(P_Q)$  has at least three solutions,
- iii) if  $Q_4 < Q$ , then  $(P_Q)$  has a unique solution, where

$$\begin{aligned} Q_1 &= \frac{(\hat{G}(-10 - \epsilon) + C_f)\rho c}{S_1 M} & Q_2 &= \frac{(\hat{G}(-10 + \epsilon) + \|f_\infty\|_\infty)\rho c}{S_0 M} \\ Q_3 &= \frac{(\hat{G}(-10 - \epsilon) + C_f)\rho c}{S_1 m} & Q_4 &= \frac{(\hat{G}(-10 + \epsilon) + \|f_\infty\|_\infty)\rho c}{S_0 m}. \end{aligned}$$

*Proof.* This proof is the extension to  $(P_{3D})$  of the results for  $(P_{2D})$  given in [40] (see also [30]). Let us define the vectorial operator  $\mathcal{A}: L^2(\Omega) \times L^2(\mathcal{M}) \rightarrow L^2(\Omega) \times L^2(\mathcal{M})$  by  $\mathcal{A}(U, u) := (AU, Bu)$  with domain

$$D(\mathcal{A}) = \{(U, u) \in L^2(\Omega) \times L^2(\mathcal{M}) : AU \in L^2(\Omega), Bu \in L^2(\mathcal{M}), U|_{\mathcal{M}} = u\},$$

where  $AU = -\operatorname{div}(\nabla U) + w \frac{\partial U}{\partial z}$  and

$$Bu = -\operatorname{div}(|\nabla_{\mathcal{M}} u|^{p-2} \nabla_{\mathcal{M}} u) + K_V \frac{\partial U}{\partial n} + F(x, \nabla_{\mathcal{M}} u) + \hat{G}(x, u).$$

It is easy to find some constant functions  $(\underline{V}, \underline{v})$  and  $(\overline{U}, \overline{u})$  verifying

$$\begin{cases} A\underline{V} = 0 \\ B\underline{v} = \frac{1}{\rho c} Q S_0 m - \|f_\infty\|_\infty \leq \frac{1}{\rho c} Q S(x) \beta(\underline{v}) + f_\infty, \end{cases} \quad \begin{cases} A\overline{U} = 0 \\ B\overline{u} = \frac{1}{\rho c} Q S_1 M - C_f \geq \frac{1}{\rho c} Q S(x) \overline{\beta}(\overline{u}) + f_\infty, \end{cases}$$

where  $\beta$  and  $\overline{\beta}$  are some (eventually discontinuous) functions (i.e., single valued sections of the graph  $\beta$ ) such that  $\beta(s) \in \beta(s)$ ,  $\overline{\beta}(s) \in \beta(s)$  and  $\beta(u) \leq h \leq \overline{\beta}(u)$  for all  $h \in \beta(u)$ . Every solution  $(U, u)$  of  $(P_{3D})$  verifies  $\underline{V} \leq U \leq \overline{U}$  and  $\underline{v} \leq u \leq \overline{u}$ .

(i) If  $Q < Q_1$  then  $\underline{V} \leq \overline{U} \leq -10 - \epsilon$ . So, every solution  $(U, u)$  of  $(P_Q)$  verifies  $u < -10 - \epsilon$  and it is a solution of the problem

$$(P_Q^m) \begin{cases} AU = 0 & \text{on } \Omega, \\ Bu = \frac{1}{\rho c} Q S(x) m + f_\infty & \text{on } \mathcal{M}, \\ U = u & \text{on } \mathcal{M}, \\ \hat{F}(x, \nabla_{\mathcal{N}} U) + \frac{\partial U}{\partial z} = 0 & \text{on } \mathcal{N}, \end{cases}$$

which has a unique solution. To prove it, we assume there exist two solutions,  $(U_1, u_1)$  and  $(U_2, u_2)$  and we take the difference  $U_1 - U_2$  as a test function in the weak formulation. The accretiveness of the operator allows us to conclude the uniqueness.

(iii) If  $Q_4 < Q$  then  $-10 + \epsilon \leq \underline{V} \leq \overline{U}$ . So, every solution  $(U, u)$  verifies  $-10 + \epsilon \leq u$  and  $\beta(u) = M$ . So,  $(U, u)$  is the unique solution of problem  $(P_Q^M)$  which is obtained by replacing  $m$  by  $M$  in problem  $(P_Q^m)$ .

(ii) The proof of the multiplicity consists of three steps

**Step 1.** Construction of upper and lower solutions. If  $Q_2 < Q < Q_3$  then,

$$\begin{aligned} \overline{U}_1 &:= \mathcal{G}^{-1}\left(\frac{1}{\rho c} Q S_1 M - C_f\right) && \text{is an uppersolution of } (P_Q^M) \\ \underline{V}_1 &:= \mathcal{G}^{-1}\left(\frac{1}{\rho c} Q S_0 M - \|f_\infty\|_\infty\right) && \text{is a lowersolution of } (P_Q^M) \\ \overline{U}_2 &:= \mathcal{G}^{-1}\left(\frac{1}{\rho c} Q S_1 m - C_f\right) && \text{is an uppersolution of } (P_Q^m) \\ \underline{V}_2 &:= \mathcal{G}^{-1}\left(\frac{1}{\rho c} Q S_0 m - \|f_\infty\|_\infty\right) && \text{is a lowersolution of } (P_Q^m). \end{aligned}$$

Moreover,  $\underline{V}_2 < \overline{U}_2 < -10 - \epsilon < -10 + \epsilon < \underline{V}_1 < \overline{U}_1$ . Then, there exist two solutions  $(U_1, u_1)$  and  $(U_2, u_2)$  of  $(P_Q)$  such that  $u_1$  and  $u_2$  do not cross the level -10. To find the third solution, we shall apply a result of [1]. This is possible for the case where  $\beta$  is a Lipschitz function. In next step, we will approximate the graph  $\beta$  by some Lipschitz functions.

**Step 2. Approximate problem.** We define a new family of problems,  $(P_{Q,\lambda})$  by replacing  $\beta(u)$  by  $\beta_\lambda(u)$  in  $(P_{3D})$ , where  $\beta_\lambda$  is the Lipschitz function  $\beta_\lambda = \frac{1}{\lambda}(I - (I - \lambda\beta)^{-1})$ ,  $\lambda > 0$  (the Yosida approximation of  $\beta$ ). Since  $\beta$  verifies  $(H_\beta)$ , we get that

$$\beta_\lambda \text{ is a bounded and nondecreasing function } \forall \lambda > 0,$$

$$\beta_\lambda(s) = \beta(s) \text{ for any } s \notin [-10 - \epsilon, -10 + \epsilon + \lambda M], \forall \lambda > 0,$$

$$\beta_\lambda(s) \rightarrow \beta(s) \text{ in the sense of maximal monotone graphs when } \lambda \rightarrow 0$$

(see [11]). In the case of  $\beta$  is a Lipschitz function, we take  $\beta_\lambda = \beta$ . Now, by applying the argument of step 1 to problem  $(P_{Q,\lambda})$ , there exit  $\lambda_0$  such that  $V_2 < \bar{U}_2 < -10 - \epsilon < -10 + \epsilon + \lambda_0 M < \underline{V}_1 < \bar{U}_1$ . Then, we have two families of solutions of  $\{(P_{Q,\lambda})\}$  such that  $u_1^\lambda$  and  $u_2^\lambda$  do not cross the level -10. We have the third family of solutions by using the following lemma. We recall that  $X$  is a retract of  $E$  if there exists a continuous mapping  $r : X \rightarrow E$  such that  $r(x) = x$  for each  $x \in X$ .

**Lemma 4.1.** (Amann [1976]) *Let  $X$  be a retract of some Banach space  $E$  and let  $F : X \rightarrow X$  be a compact map. Suppose that  $X_1$  and  $X_2$  are disjoint retracts of  $X$ , and let  $Y_k$ ,  $k = 1, 2$  be open subset of  $X$  such that  $Y_k \subset X_k$ . Moreover, suppose that  $F(X_k) \subset X_k$  and that  $F$  has no fixed points on  $X_k - Y_k$ ,  $k = 1, 2$ . Then  $F$  has at least three distinct fixed points  $x, x_1, x_2$  with  $x_k \in X_k$  and  $x \in X - (X_1 \cup X_2)$ .*

We see that the assumptions of this lemma are satisfied. Any solution  $u$  of the problem  $(P_{Q,\lambda})$  is a fixed point of the equation  $u = F(u)$  with  $F : L^\infty(\mathcal{M}) \rightarrow L^\infty(\mathcal{M})$  defined by

$$u = P_2(\mathcal{A}^{-1}(\frac{1}{\rho c}QS(\cdot)\beta_\lambda(u) + f_\infty(\cdot))),$$

where  $P_2$  is the projection over the second component. Let  $E = L^\infty(\mathcal{M})$  which is an ordered Banach space with respect to the natural ordering whose positive cone is given by

$$L_+^\infty(\mathcal{M}) = \{v \in L^\infty(\mathcal{M}) : v(x) \geq 0 \text{ a.e. } x \in \mathcal{M}\},$$

having a nonempty interior. Let us define the intervals  $X = [\underline{V}_2 - \delta, \bar{U}_1 + \delta]$ ,  $X_1 = [\underline{V}_1 - \delta, \bar{U}_1 + \delta]$  and  $X_2 = [\underline{V}_2 - \delta, \bar{U}_2 + \delta]$  where  $\delta > \lambda_0 M$  is taken such that  $\underline{V}_1 > -10 + \epsilon + \delta$ ,  $\bar{U}_2 > -10 - \epsilon - \delta$ . So, there exists an open set  $Y_k$  of  $L^\infty(\mathcal{M})$  containing  $u_k^\lambda$  for  $k = 1, 2$  such that  $Y_k \subset X_k$ . The sets  $X, X_1$  and  $X_2$  are retracts of  $L^\infty(\mathcal{M})$  (resp.  $X$ ), since they are nonempty closed convex subsets of  $L^\infty(\mathcal{M})$  (resp.  $X$ ). Moreover,  $F(X) \subset X$  and  $F(X_k) \subset X_k$ . Finally, from the properties of  $\beta_\lambda$  and the compact embedding  $W^{1,p}(\mathcal{M}) \subset L^\infty(\mathcal{M})$  for  $p \geq 2$ , we arrive to  $F : X \rightarrow X$  is a compact map. So, by Lemma 4.1 we conclude that  $F$  has at least three fixed points, or equivalently,  $(P_{Q,\lambda})$  has at least three solutions:  $u_1^\lambda \in X_1$ ,  $u_2^\lambda \in X_2$  and  $u_3^\lambda \in X - (X_1 \cup X_2)$ .

**Step 3.** The proof ends with the convergence of a subsequence of  $\{u_3^\lambda\}$  to  $u_3$  such that  $(U_3, u_3)$  is a solution of  $(P_{Q,\lambda})$ . To get this limit we need to use a result of maximal monotone graphs ([10]) which guarantees that the limit of  $\beta_\lambda(u_\lambda)$  is in the graph  $\beta(u_3)$ . Finally, the convergence in  $L^\infty(\mathcal{M})$  allow us to show that  $u_3$  is different from  $u_1$  and  $u_2$ . In particular,  $u_3$  must cross the level -10.

## 5. Numerical approximation

In this section we are concerned with computing a numerical solution for the problem  $(P_{2D})$  with  $p = 3$ . The numerical approximation used is based upon the finite volume method with Weighted Essentially Non-Oscillatory (WENO) reconstruction in space and third-order Runge-Kutta TVD for time integration. Details of WENO reconstruction can be found in many references, among them [5], [65], [66], [43]. For each time step, a numerical solution of the EBM is computed and then used as a Dirichlet boundary condition for the deep ocean model. Other approximations are possible, for instance we can mention the ADER-ENO scheme for nonlinear reaction-diffusion problems proposed by [67]. The numerical scheme follows the ideas put forward in [51]. Its application allows to obtain  $\gamma_{i,j}^{n+1}$  for each control volume. Then we use an iterative solver of

**Table 1.** Physical data used in the model.

<i>Parameter</i>	<i>Value</i>	<i>Units</i>
$K_H$	0.049	$m^2 c^{-1}$
$K_{H_0}$	$0.555 \times 10^{-3}$	$m^2 c^{-1}$
$K_V$	0.0125	$m^2 c^{-1}$
$C, B$	190, 2	$W m^{-2}, W m^{-2} K^{-1}$
$c, \rho$	3900, 1004	$J (kg^\circ C)^{-1}, kg m^{-3}$
$Q$	340	$W m^{-2}$
$D$	60	$m$

nonlinear equations to compute the cell averages of the numerical solution for the deep ocean model  $U_{i,j}^{n+1}$  from  $\gamma_{i,j}^{n+1}$ , solving the nonlinear equation

$$\gamma_{i,j}^{n+1} = \begin{cases} k_1 U_{i,j}^{n+1,iter} & \text{if } U_{i,j}^{n+1,iter-1} < 0 \\ \frac{L+k_2\epsilon}{\epsilon} U_{i,j}^{n+1,iter} & \text{if } 0 \leq U_{i,j}^{n+1,iter-1} < \epsilon \quad (iter = 1, 2, \dots), \\ k_2 U_{i,j}^{n+1,iter} + L & \text{if } U_{i,j}^{n+1,iter-1} \geq \epsilon \end{cases}$$

for a given small  $\epsilon$ . This iterative process ends up when  $|U_{i,j}^{n+1,iter} - U_{i,j}^{n+1,iter-1}| < \delta$  for each control volume  $V_{i,j}$  and with  $\delta$  small enough. The iterative solver used consists of a combination of Newton's method and bisection method, in such a way that the method performing is the one that converges faster. Note that, following this idea, both methods can act at a particular time step. Finally, we assign the value  $U_{i,j}^{n+1} = U_{i,j}^{n+1,iter}$ . As for the cell averages of the delay term  $u_i(t - \tau)$ , an arithmetic mean of the values  $u_i(t^k)$  and  $u_i(t^{k+1})$  with  $t - \tau \in [t^k, t^{k+1}]$  has been used.

The evolution of the temperature in the deep ocean is due to the combined effect of water sinking from the Earth poles with heating-cooling processes taking place in the interface atmosphere-ocean. Also water *upwelling* takes place at certain latitudes.

In the first numerical example we compare the numerical solution of the model with and without the effect of the latent heat. The initial conditions considered are  $U(0, x, z) = 18e^{-x^2-z^2} + 6e^{6z}(11e^{-x^2} - 10)$  for the ocean interior and  $u(0, x) := U(0, x, 0) = 84e^{-x^2} - 60$ . The data used in this example are depicted in Table 1. In this table the unit  $c$  stands for century. The insolation function is taken as  $S(x) = 1 - \frac{1}{2}P_2(x)$  where  $P_2(x) = \frac{1}{2}(3x^2 - 1)$  is the second Legendre polynomial in the interval  $[-1, 1]$ . The coalbedo  $\beta(u)$  is given by  $(H_1^*)$ , where  $m = 0.4$  and  $M = 0.69$ . The numerical implementation of the coalbedo is performed considering that we are in the context of an explicit scheme therefore if, in the previous time step, at certain control volume,  $u \leq -10$  then  $\beta(u) = m$  otherwise,  $\beta(u) = M$ . As for the velocity, it depends only on  $x$  and in this work it is defined as

$$\omega(x, z) = W(x) = \frac{10(x + 0.75)(x - 0.75)}{(0.1 + 10|x + 0.75|)(0.1 + 10|x - 0.75|)}. \quad (5.1)$$

This particular velocity is a way to represent sinking water near the poles and *upwelling* water in the vicinity of the Equator. The spatial discretization used is  $\Delta x = 2/60$ ;  $\Delta z = 1/60$  and the size of the time step is calculated in an iterative way according to the formula

$$(\Delta t)_n = \min(\alpha \Delta x^2 ((1 - x^2)K_H)^{-1}, \alpha \Delta z^2 (K_V)^{-1}, \alpha \Delta x^2 ((1 - x^2)K_{H_0} \left| \frac{du_{n-1}}{dx} \right|)^{-1}), \quad (5.2)$$

where  $\alpha = 0.3$  for stability reasons. Other values used here are  $k_1 = k_2 = 1$ ,  $\epsilon = 0.01$  and  $L = 3$ . The numerical experiment with latent heat shows more clearly (than the experiment without this term) the crucial role of the deep ocean: indeed, besides a suitable justification of the formation of sea ice sheets (the level lines of the lower values of the sea temperature are now more separated,

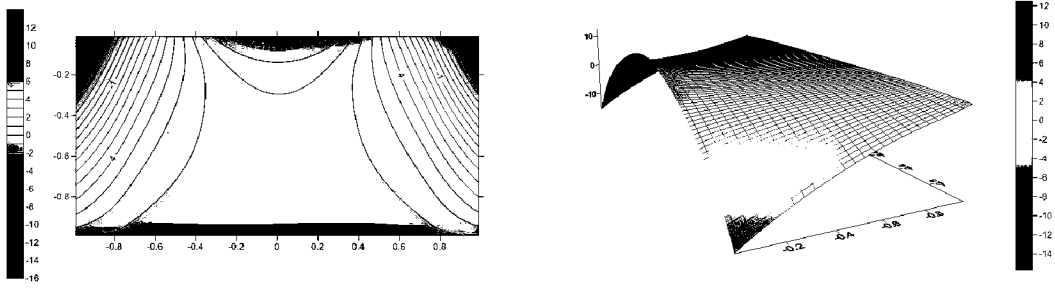


Figure 1. Temperature without latent heat ( $\gamma(u) = u$ ) and  $t = 5$ .

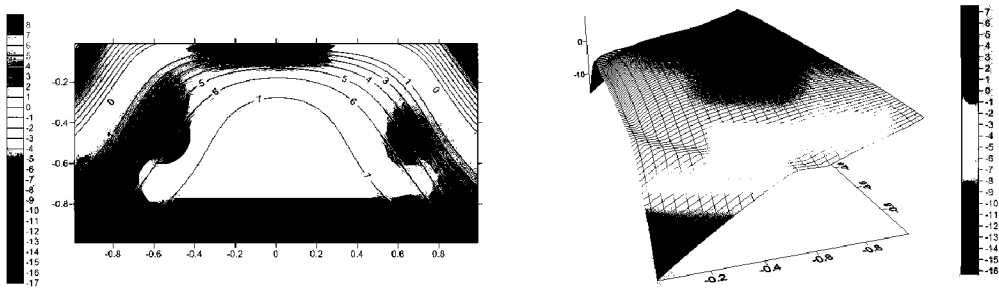


Figure 2. Temperature with latent heat and  $t = 5$ .

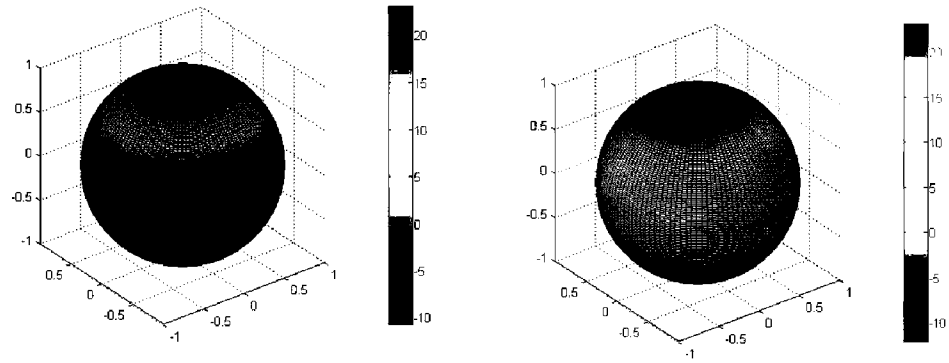


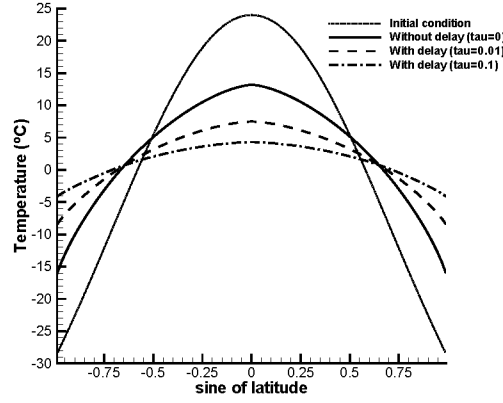
Figure 3. Mean Surface Temperature without latent heat effect (left) and with latent heat effect (right) at  $t = 4$ .

which corresponds to the presence of large regions without great abrupt temperature changes), most of the higher level lines of the sea temperature does not arrive to touch the sea surface (except, at most, some of them which do that around the Equator).

We can generate solutions of  $(P_{3D})$  from the solutions of  $(P_{2D})$  under suitable conditions. In Figure 3 we can see the distribution of temperature on the Earth surface obtained by the numerical approximation of  $(P_{2D})$  and rotated thanks to the spherical coordinates. We observe that the surface temperature is lower in the case of presence of latent heat than when this effect is neglected. This is precisely what may be considered as an alarm about the gravity of the global change, since if a *realistic* deep ocean (that means with latent heat) is heated, the time to return to previous colder situations may be very large.

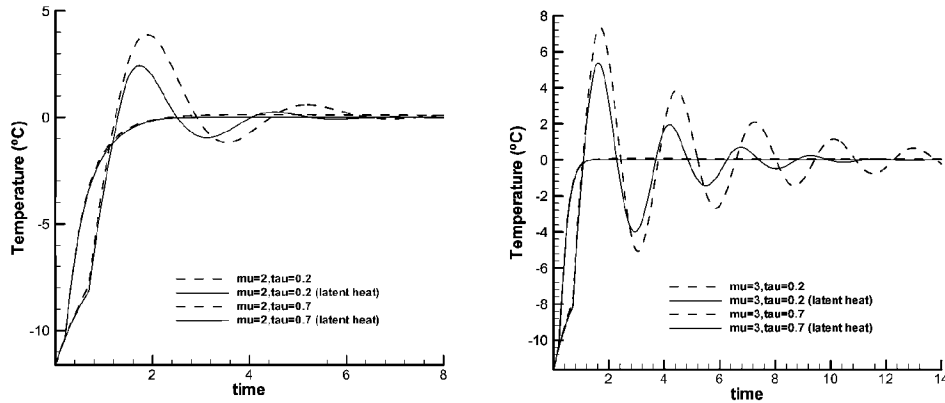


A numerical experiment carried out considers the delay effect. The results can be seen in Figure 4. The range of temperatures is more narrow when considering this term than without its influence. Therefore the delay term is like a memory one, which *remembers* the temperature of previous time steps and, therefore, tends to smooth the spatial evolution of the temperature. In this example we have taken  $\mu = 0.5$ , but no latent heat, for simplicity. Another interesting feature



**Figure 4.** Delay effect in the upper boundary (without latent heat effect) at  $t = 5$  for different values of the time of delay  $\tau$ .

of the effect of the delay term is depicted in Figure 5 where the temperature is plotted as a function of time for the particular latitude  $38^\circ S$  and for different values of the parameters  $\mu$  and  $\tau$ . The results show that, in both situations, a stationary state is reached. Nevertheless, when the time of delay  $\tau$  is larger the solution becomes more oscillating and takes a longer time in reaching the stationary state. This effect is more evident for larger values of  $\mu$ . This conclusion is similar to that pointed out in [58]. Also Figure 5 reveals that the consideration of the latent heat effect give rise to less oscillating solutions.



**Figure 5.** Temperature in the upper boundary as a function of time, with (full lines) and without (dashed lines) latent heat effect, for the latitude  $38^\circ S$ , being  $\mu = 2$  (left) and  $\mu = 3$  (right) and, in both cases, for two different values of the time delay  $\tau$ .

*Acknowledgements.* The research of JID and LT was partially supported by the project MTM2008-06208 and MTM2011-26119 (DGISPI, Spain). The research of JID has received funding from the ITN FIRST of the Seventh Framework Programme of the European Community (grant agreement number 238702) and the Research Group MOMAT (Ref. 910480) supported by UCM. The research of the second author was partially supported by the project Consolider-Ingenio 2010 CSD2009-00065.

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